Elastic, Cottage Cheese, and Gasoline:

Visualizing Division of Fractions

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Representation is one of the five Process Standards from *Principles and Standards for School Mathematics.* This Standard proposes that prekindergarten through grade 12 instructional programs should enable students to "select, apply, and translate among mathematical representations to solve problems" (NCTM 2000, p. 67). To implement a program that meets this Standard, teachers themselves must have attained this level of proficiency.

According to Liping Ma (1999, p. 55), "Division by fractions, the most complicated operation with the most complex numbers, can be considered as a topic at the summit of arithmetic." During her research, only 39 percent of the 23 teachers in the United States she surveyed were able to find a correct answer when calculating 1 3/4 ÷ 1/2. Conversely, 100 percent of the 72 Chinese teachers in the study found the correct answer. When the task progressed from computation to representation of the division problem, U.S. teachers fared even worse; only 1 of the U.S. teachers was able to construct a representation (verbal or visual) that was conceptually correct. Of the Chinese teachers, 90 percent were able to produce conceptually correct representations for this division problem.

Where do you stand? The answer might surprise you. Before reading further, take time to create a word problem that would be solved using the equation 2 2/3 ÷ 2/3. Then create a diagram or use manipulatives to model the situation.

ELASTIC: LOUIS AGUINALDO/ISTOCKPHOTO.COM; COTTAGE CHEESE: DAVE WHITE/ISTOCKPHOTO.COM; GASOLINE: DANNY HOOKS/ISTOCKPHOTO.COM

THE WORKSHOP

This article describes the challenges that one group of teachers faced in representing division problems. The problems were encountered at a Teacher Leader Academy produced by the Math Science Partnership of southwestern Pennsylvania, a National Science Foundation-funded program to improve the teaching of mathematics and science at the K–12 grade levels. This weeklong workshop focused on helping elementary school and middle school teachers deepen their understanding of the four arithmetical operations.

The Teacher Leader Academies are organized by mathematics coordinators from the Allegheny Intermediate Unit (an educational resource facility that serves elementary and secondary schools in the Pittsburgh region). The coordinators spend most of the year collecting and developing high-quality workshop material from a variety of sources. During the summer months, they meet with teacher leaders at area colleges and universities and prepare them to plan and produce teacher professional development programs in their own districts. University faculty are also involved in this process as planners and workshop facilitators.

The mathematics coordinators selected material for one session from *Numbers and Operations: Making Meaning for Operations* (*Part 2*) of the Developing Mathematical Ideas series by Dale Seymour Publications (Schifter, Russell, and Bastable 1999).

The Elastic Stretch Problem

On the last day of the workshop, the participants considered a case study that posed this problem:

> A piece of elastic can be stretched to 5 1/2 times its original length. When fully stretched, it is 33 meters long. What was the elastic's original length? (Schifter, Russell, and Bastable 1999, chap. 7)

Fig. 1 The first visualization of stretched elastic

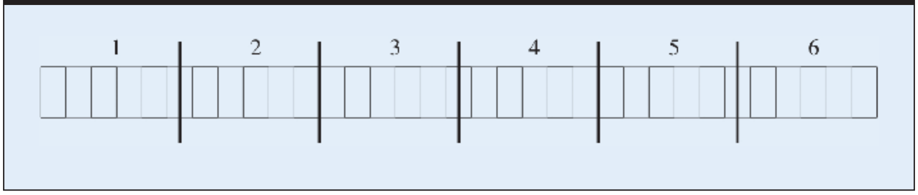


Fig. 2 The second visualization of stretched elastic

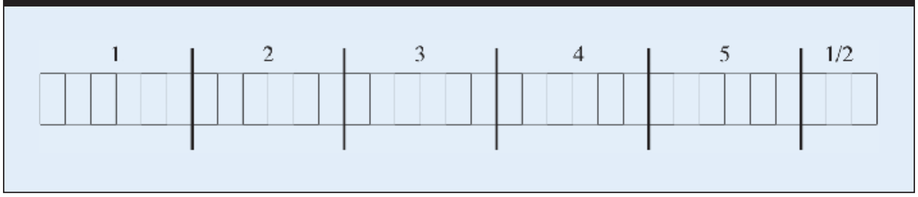
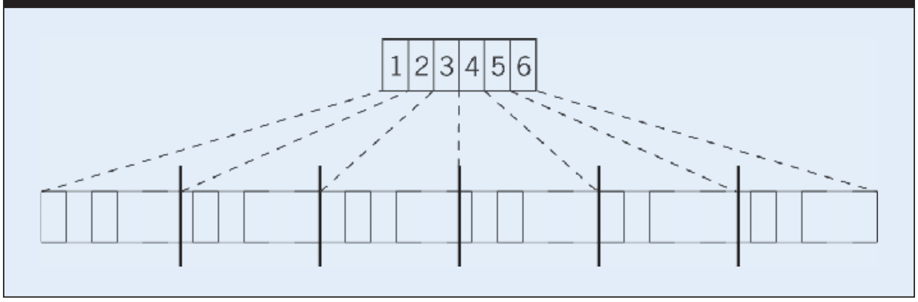


Fig. 3 An alternate visualization of stretched elastic



A student in the case study described his group's solution by drawing 33 rectangles and stating that they wanted to put the rectangles into 5 1/2 groups. However, the group produced a diagram that showed the 33 meters divided into 6 parts (see **fig. 1**), not 5 1/2 parts (Schifter, Russell, and Bastable 1999, chap. 7).

How did the students in the case study find the first diagram? The consensus in our workshop was that the students realized that the Elastic Stretch problem is solved by the division problem 33 ÷ 5 1/2. They represented this associated division problem without connecting it to the context of the original problem. After extensive discussion and more student work, another group of students in the case study produced the diagram in **figure 2** (Schifter, Russell, and Bastable 1999, chap. 7) that shows 5 1/2 groups of 6 meters each.

The answer to the stretch problem is that the elastic originally measured 6 meters. The students in the case study, as well as the participants in the Teacher Leader Academy, agreed that the second figure is a better representation of the Elastic Stretch problem because it shows 33 meters partitioned into 6 meters 5 1/2 times. **Figure 1** shows 6 groups, not 6 meters.

Then one of the participants, Sallie Peck, spoke up, "What if the 6 parts of the first picture each represent 1 meter of the original elastic that has been stretched to 5 1/2 meters?" Peck drew 6 squares above the first picture and connected them to the original diagram (see **fig. 3**), showing a before and after situation. Understanding student errors is an important skill for assessing student work. In this case, Peck had to use a similar skill to understand student work that was possibly correct.

Researchers typically classify multiplication and division problems into four classes. The two most common classes are called *equal groups* and *multiplicative comparison* problems (Cathart et al. 2003; Van de Walle 2004). These classifications are called *sharing/measurement* and *ratio quotient* in Rubenstein, Beckmann, and Thompson (2004). The students in the case study agreed on an *equal groups* interpretation of the problem, in which the entire 33 meters of stretched elastic are partitioned into 5 1/2 equal groups of 6 meters of unstretched elastic. The students later demonstrated that they interpreted the Elastic Stretch task as a measurement problem (also in the *equal groups* class of problems) by subtracting 5 1/2 six times from 33:

$$33-5\frac{1}{2}-5\frac{1}{2}-5\frac{1}{2}-5\frac{1}{2}-5\frac{1}{2}-5\frac{1}{2}=0$$

Peck was able to see the situation as being a *multiplicative comparison* problem. Van de Walle describes this as follows: "There are really two different sets, . . . one set consists of multiple copies of the other" (2004, p. 144). In the Stretched Elastic problem the two sets are (1) the number of meters of unstretched elastic and (2) the number of meters of stretched elastic. Each meter of unstretched elastic corresponds to 5 1/2 meters of stretched elastic.

Classifying the multiple interpretations for the students is usually not the teacher's goal. It is crucial, however, that the classroom teacher recognizes that they are valid strategies and that he or she must be flexible when facilitating alternate approaches. Peck's solution showed that unexpected alternate strategies are sometimes valid—with some work. It is essential that teachers encourage their students to go beyond computation. Students and teachers alike must be able to explain the mathematics and express the situation with symbols, charts, graphs, and diagrams, which are all ways of communicating mathematically. Creating these representations and interpreting the representations of others require that students make connections. "Students who are more comfortable when performing operations than they are when reasoning fail to make connections and are unable to transfer information to new situations, even when the transfer represents only a small step" (NCTM 2000).

Teachers realized the importance of being able to conceptually represent the mathematics so that they can guide their students to create and understand alternative representations

"Moving from one representation to another is an important way to add understanding to an idea. As teachers help develop flexibility with a variety of representations for mathematical ideas, students not only add to their own understanding but also acquire skill in applying mathematical ideas to new areas and communicating ideas to others" (Van de Walle 2004, p. 5). When students can create and identify corresponding representations, their understanding of relationships will be extended (Chapin and Johnson 2000). We cannot adequately develop this flexibility in our students if we have not acquired it ourselves.

The Cottage Cheese and Lawn Mower Problems

The facilitator of the workshop shared these follow-up questions with participants.

1. I eat 2/3 cup of cottage cheese for lunch each day. I have 2 2/3 cups of cottage cheese in my refrigerator. How long will that last me?
2. I put 2 2/3 gallons of gas into my empty lawn mower. I notice that it is now 2/3 filled. What is the capacity of my gas tank? (Schifter, Russell, and Bastable 1999, p. 69)

Earlier in this article, a question of representation for the division problem 2 2/3 ÷ 2 was posed. Was your representation anything like these situations?

The following description allows us to classify these representations:

> When the number and size of groups are known, the problem is a multiplication situation. When either the number of sets or the size of sets is unknown, division results. But note that these latter two situations are not alike. Problems where the size of the sets is unknown are called *fair-sharing* or *partition* problems.
>
> The whole is shared or distributed among a known number of sets to determine the size of each. If the number of sets is unknown but the size of the equal sets is known, the problems are called *measurement* or sometimes *repeated-subtraction* problems. The whole is "measured off" in sets of the given size. (Van de Walle 2004, p. 144)

The Cottage Cheese task is a *repeated-subtraction,* or *measurement,* problem, because we must determine the number of days that we consume 2/3 cups of cottage cheese. The Lawn Mower task is harder to classify, but we can make a case that it is a *fair-sharing,* or *partition,* problem. The

Fig. 4 An accepted visualization of the Lawn Mower problem

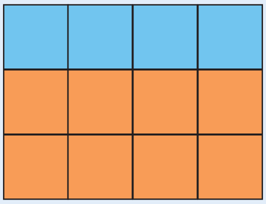
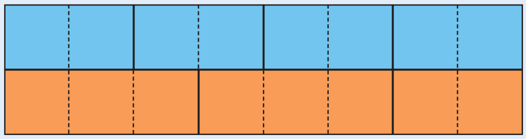


Fig. 5 How can this diagram represent the Lawn Mower problem?



gas tank's capacity is unknown, and each gallon of capacity should hold an equal amount (2/3 of a gallon) of gas for a total of 2 2/3 gallons of gas.

The facilitator asked the participants to find a multiplication sentence and a division sentence for each problem and to model each sentence with a diagram or manipulatives. The workshop participants had no trouble agreeing on a suitable representation for the first problem (and perhaps the reader will pause to construct her or his own model before continuing on). The second problem turned out to be much more difficult.

Because the problem was challenging, the discussion became intense. The participants were not able to agree on any sort of visualization of either the multiplication $4 \times 2/3 = 2\ 2/3$ or the division $2\ 2/3 \div 2/3 = 4$ that would accurately represent the context of the problem. This problem was much more difficult than anticipated. The facilitator drew two representations for the Lawn Mower problem on the board. The first representation shows a 3×4 rectangle composed of blue and orange squares (see **fig. 4**). One teacher quickly explained that if each square represents 1/3 gallon of gasoline, then the orange squares total 2 2/3 gallons of gasoline arranged in columns, where 2/3 of each column is "filled" with gasoline. Since there are 4 columns, the capacity of the gas tank must be 4 gallons. The blue squares in the top row each represent 1/3 of each gallon of empty space (or air), which completes the full gas tank.

Table 1 The Lawn Mower problem as a rate problem

Contents	Full Capacity
2/3	1
1 1/3 = 4/3	2
2 = 6/3	3
2 2/3 = 8/3	4

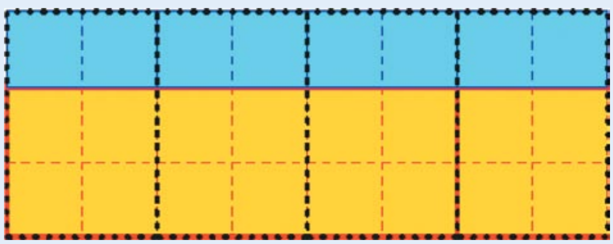
The second representation shows two rows of blocks (see **fig. 5**). The bottom row is made of 1×3 orange blocks, specifically, 2 2/3 of them. The top row is made of four 1×2 blue blocks aligned directly over the bottom row. The teachers agreed that this figure clearly represented the problem $2\ 2/3 \div 2/3 = 4$, since the total of 2 2/3 is *measured* by four blue blocks representing 2/3 of each orange box.

The participants ran into a problem when the facilitator asked them to connect the diagram with the context of the Lawn Mower problem. They did not think this was possible. Their objection was that if each orange 1×3 rectangle represents 3/3 gallons of gas, then how could each *smaller* blue rectangle also represent a gallon? Peck resolved the confusion by interpreting the second problem as a *rate* problem. Since the gas tank is 2/3 full, each 2/3 of a gallon of gas that was put into the tank corresponds to 1 gallon of capacity. She drew **table 1**, which shows how each blue rectangle could simultaneously represent 2/3 gallon of gas *and* correspond to a full gallon of tank capacity.

To present additional insight, the facilitator added a row of orange blocks, so that a full gas tank was represented by six squares to the gallon; 2/3 of each gallon, shown in orange, is filled with gasoline. This is demonstrated by outlining four groups of six squares, as in **figure 6**. This shows that the gas tank is 2/3 full and 1/3 empty and has a capacity of 4 gallons, each of which is outlined in black. The gas tank contains 2 2/3 gallons of gas, shown in orange.

The participants agreed that by creating, interpreting, and discussing the various representations, they now had a better understanding of fraction division. They realized the importance of teachers being able to conceptually represent the mathematics so that they can guide their students to create and understand alternative representations. Each teacher was confident that

Fig. 6 Visualizing the gas tank



the activities greatly enhanced their ability to help their students develop flexibility with representations, and, ultimately, deepen their understanding of mathematics.

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